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Pi Networks with Loss

Inductor losses can substantially alter the behavior of a pi network and place strong constraints on the overall circuit quality factor.

In a recent *QEX* paper[1] I examined the relation between the quality factor Q and bandwidth of pi networks. Several definitions of Q that appeared in the *ARRL Handbook for Radio Communications*[2] were not good predictors of the bandwidth of pi networks, but a revision of one did predict bandwidth, especially for values of Q greater than about 5. That paper assumed that any loss in the pi networks under consideration could be neglected.

After the publication of the paper, I received an e-mail communication suggesting that I look at the loss in the inductor used in pi networks. I decided to extend my analysis to include inductive loss. Several months later, I finished this analysis, which turned out to be considerably more difficult than I expected. The results were interesting, and in some cases striking. Inductive loss can substantially alter the behavior of a pi network and place quite strong constraints on the overall circuit quality factor.

There seems to be general agreement that losses in the inductors in pi networks are more significant than those in the capacitors. I discovered a paper online[3] which found that the Q's for typical air variable capacitor are typically 500 or more. Q's for practical inductors are considerably less. Consequently, I limited my analysis to inductor loss.

The following sections of the paper develop equations for the values of the various components of a pi network and the Q and internal loss in this network. I also measured the actual loss of the rotary variable inductor in my antenna tuner and used these data to demonstrate how a pi network could be designed taking into account loss within the network itself.

Analysis of a Pi Network Including Inductive Loss

Figure 1 shows a pi network. Loss in the inductor is represented by the inductor's "equivalent series resistance," labeled R_2. R_2 will be referred to as the loss resistance in the remainder of this paper. Initially, the loss resistance will be assumed constant, regardless of the value of the inductance. This is a substantial limitation because, in reality, R_2 will always vary as the inductance of L_2 varies. At the end of this paper I will describe how data relating R_2 to L_2 can be used to overcome this limitation.

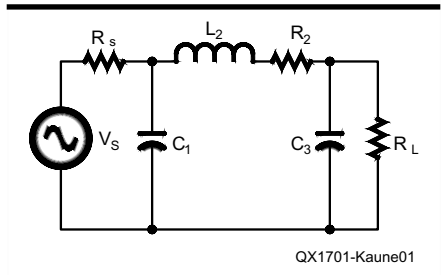

Figure 1 — Schematic of pi network showing R_2, the loss resistance of the inductor.

Pi networks are used frequently in power amplifiers — especially those using electron tubes — to match a load resistance — often an antenna — to a much larger source resistance. Load matching is accomplished when the impedance looking into the input of the pi network equals the source impedance. Here, this condition means that that the resistance looking into the input must be equal to the source resistance, R_S, and the reactance looking into the input must be 0. To achieve these two requirements, we have three quantities that we can adjust, C_1, L_2, and C_3. Thus, to uniquely specify the capacitances and the inductance, we need a third parameter to be specified. In my earlier paper the reactance of C_1 was used as this parameter. However, I found after some experimentation that, when inductive loss is included, simpler results were obtained by dividing the pi network into two cascaded

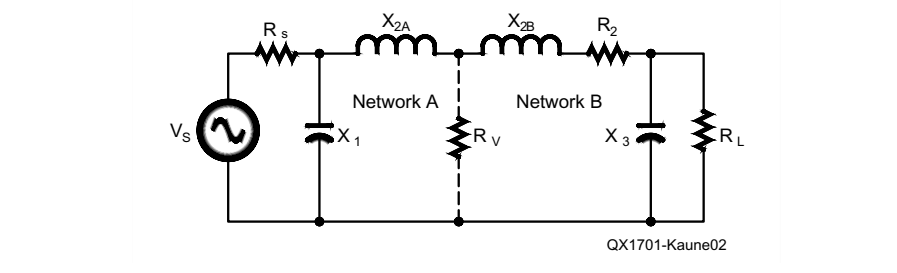

Figure 2 — Pi network divided into two cascaded sub-networks, *A* and *B*. The load for network *A* is a virtual resistance R_V, which is also the source for Network *B*. Network *A* is adjusted to match R_V to R_S, and Network *B* is adjusted to match R_L to R_V.

L-type networks with a virtual resistance, R_V, as the matched load of the first and the source of the second. R_V then becomes the third parameter needed to uniquely specify the components that are labeled by their reactances in Figure 2.

In Figure 2, the two sub-networks are *A* and *B* and the inductor L_2 (with reactance X_2) has been divided into two parts, L_{2A} (X_{2A}) and L_{2B} (X_{2B}). For a given value of R_V, unique values of X_1 and X_{2A} can always be found that match R_V to the source resistance R_S, provided that R_V satisfies certain criteria that will be derived shortly. And, X_{2B} and X_3 can also always be adjusted to obtain a match between R_L and R_V as long as R_V satisfies additional constraints.

Consider then Network *A* shown in Figure 3. This network can be simplified by replacing the series combination of X_{2A} and R_V with their parallel equivalents, X'_{2A} and R'_V, respectively. The parallel-equivalent quantities are related to the series value by the following well-known equations:

$$R'_V = \frac{R_V^2 + X_{2A}^2}{R_V} \tag{1}$$

and

$$X'_{2A} = \frac{R_V^2 + X_{2A}^2}{X_{2A}}. \tag{2}$$

This reconfigured network is shown in Figure 4. It has become a simple parallel *RLC* network. In order for the load, R'_V, to be matched to the source, the following equations must be satisfied:

$$R_S = R'_V \tag{3}$$

and

$$X_1 + X'_{2A} = 0. \tag{4}$$

Now use Eq. (1) to eliminate R'_V in Eq. (3) and Eq. (2) to eliminate X'_{2A} in Eq. (4), and solve the resulting two equations for X_{2A} and X_1. The results are

$$X_1 = -R_S\sqrt{\frac{R_V}{R_S - R_V}}, \tag{5}$$

$$X_{2A} = \sqrt{R_V\left(R_S - R_v\right)}. \tag{6}$$

Both X_1 and X_{2A} must be real numbers, which requires that

$$R_V < R_S. \tag{7}$$

Turn to Network *B* in Figure 2. With R_V matched to R_S in Network *A*, its Thevenin equivalent is an ideal voltage source, V_A, in series with a resistance R_V. Thus, Network *B* can be diagrammed in Figure 5. Next convert the parallel pair of X_3 and R_L to an equivalent series pair, X'_3 and R'_L, where

$$R'_L = R_L\frac{X_3^2}{R_L^2 + X_3^2}, \tag{8}$$

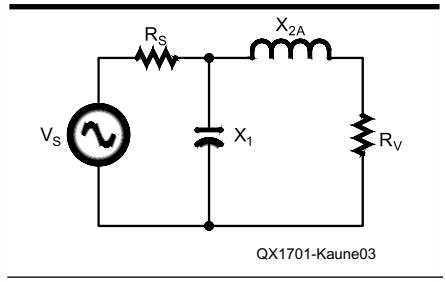

Figure 3 — Schematic of Network *A*.

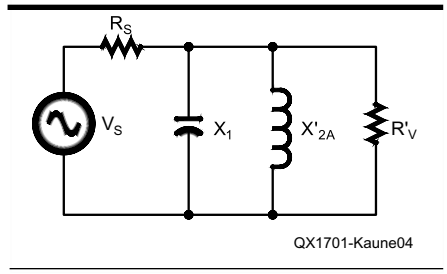

Figure 4 — Network *A* with the series combination of X_{2A} and R_V shown in Figure 3 replaced with their parallel equivalents X'_{2A} and R'_V.

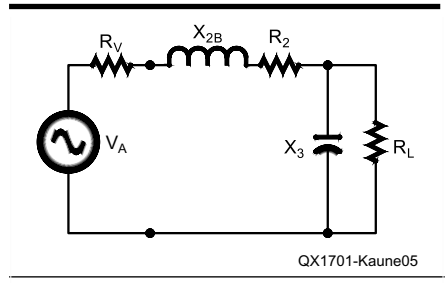

Figure 5 — Schematic of Network *B* with Network *A* replaced with its Thevenin equivalent of an ideal voltage source V_A in series with a resistance R_V.

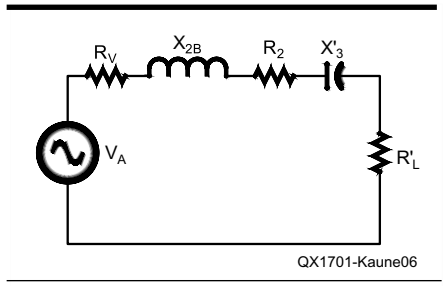

Figure 6 — Network *B* with the parallel combination of X_3 and R_L shown in Figure 5 replaced with its series equivalent X'_3 and R'_L.

and

$$X'_3 = X_3\frac{R_L^2}{R_L^2 + X_3^2}. \tag{9}$$

The resulting simple series *RLC* network is shown in Figure 6. The load is R_2 and R'_L in series. To match this load to the source resistance R_V the following two equations must be satisfied:

$$R_V = R_2 + R'_L, \tag{10}$$

and

$$X_{2B} + X'_3 = 0. \tag{11}$$

Use Eq. (8) to eliminate R'_L in Eq. (10), and Eq. (9) to eliminate X'_3 in Eq. (11). Then solve the resulting two equations for X_{2B} and X_3. The results are:

$$X_{2B} = \sqrt{\left(R_V - R_2\right)\left(R_L + R_2 - R_V\right)} \tag{12}$$

$$X_3 = -R_L\sqrt{\frac{R_V - R_2}{R_L + R_2 - R_V}}. \tag{13}$$

X_{2B} and X_3 must both be real numbers, which requires that

$$R_2 \le R_V < R_L + R_2. \tag{14}$$

The results so far are below:

$$\left.\begin{aligned} X_1 &= -R_S\sqrt{\frac{R_V}{R_S - R_V}} \\ X_2 &= \sqrt{R_V\left(R_S - R_v\right)} \\ &\quad + \sqrt{\left(R_V - R_2\right)\left(R_L + R_2 - R_V\right)} \\ X_3 &= -R_L\sqrt{\frac{R_V - R_2}{R_L + R_2 - R_V}} \\ R_2 &\le R_V < \mathrm{Min}\left(R_S, R_L + R_2\right) \end{aligned}\right\} \tag{15}$$

Quality Factor and Loss

The quality factor Q of a network is defined by the following equation.

$$Q = 2\pi f\frac{\text{Energy stored in network}}{\text{Power dissipated in network}}, \tag{16}$$

where f is frequency. There are several definitions of Q in use, depending on which contributors to loss are included in the denominator. Since in this paper we are concerned with the behavior of the network (i.e. its bandwidth and harmonic attenuation) when it is used to match a load to a source, we will include all sources of power loss including the source and load resistances.

To calculate the Q of a pi network, first calculate the quality factors, Q_A and Q_B, of Networks A and B, respectively. Network A (Figure 3) was transformed into a simple parallel *RLC* network (Figure 4). The quality factor of this network is the net resistance in the network divided by the inductive (or capacitive) reactance. As shown in Figure 4, the resistors R_S and R'_V are in parallel because the resistance of the ideal voltage source V_S is 0. But, Eq. (3) states that the two resistances are equal, so the net resistance is $R_S/2$. Consequently,

$$Q_A = \frac{R_S}{2|X_1|} = \frac{1}{2}\sqrt{\frac{R_S - R_V}{R_V}}, \qquad (17)$$

where Eq. (5) has been used to eliminate $|X_1|$.

Network B (Figure 5) was transformed into a simple series *RLC* network (Figure 6). The quality factor for this network is the reactance of the inductor (or capacitor) divided by the net resistance, $R_V + R_2 + R'_L$, in the circuit. But Eq. (10) states that $R_V = R_2 + R'_L$, so

$$Q_B = \frac{X_{2B}}{2R_V} = \frac{\sqrt{(R_V - R_2)(R_L + R_2 - R_V)}}{2R_V}, \qquad (18)$$

where Eq. (12) has been used to eliminate X_{2B}.

What is the composite Q of the cascaded network consisting of Networks A and B given Q_A and Q_B, the Q's of the component sub networks? I treated this problem [Note 1] for the case where the two networks were lossless; the overall Q was obtained by simply adding the individual Q values of the constituent networks. In Appendix A, the case where the network has loss is analyzed, with the same result that the Q for the overall network is still obtained by adding the Q values for the constituent parts. (This is true only if the network loss occurs only in Network B. This is why I placed the loss resistor, R_2, in Network B. R_2 could have been placed in Network A, or it could have been divided in any way between the two networks with equivalent results.) Consequently, the system Q for the entire pi network is

$$Q = \frac{1}{2}\left[\sqrt{\frac{R_S - R_V}{R_V}} + \sqrt{\left(\frac{R_V - R_2}{R_V}\right)\left(\frac{R_L + R_2 - R_V}{R_V}\right)}\right] \qquad (19)$$

We now calculate the loss in the pi network. The loss, of course, occurs in the inductor loss resistance, R_2. Since this resistance was placed entirely in Network B, there is no loss in Network A. Referring to Figure 6, we see that the current through the load, R'_L, is the same as the current through R_2. Consequently, the powers, P_L and P_2, dissipated in the load and inductor resistance, respectively, are proportional to R'_L and R_2. The efficiency, ε, of the pi network to deliver power to its load is

$$\varepsilon = \frac{P_L}{P_2 + P_L} = \frac{R'_L}{R_2 + R'_L}. \qquad (20)$$

According to Eq. (10), $R'_L = R_V - R_2$. Using this to eliminate R'_L in Eq. (20), we arrive at the remarkably simple result

$$\varepsilon = 1 - R_2/R_V. \qquad (21)$$

It is conventional to express loss in decibels. Denote this quantity L_{dB}. Then,

$$L_{dB} = -10\log_{10}(1 - R_2/R_V). \qquad (22)$$

All of the quantities in Eqs. (15), (19), and (22) are functions of the parameter R_V. In a later section we address the question of how to select values of R_V to achieve either a desired Q or loss for a pi network.

Relationship between *Q*, Bandwidth, and Harmonic Attenuation

Previously (Note 1), I found that the following definition provided an accurate prediction of lossless network bandwidth and harmonic attenuation:

$$Q = \frac{1}{2}\left(\frac{R_S}{|X_1|} + \frac{R_L}{|X_3|}\right). \qquad (23)$$

For a pi network with loss, Q is given by Eq. (19) which appears different than Eq. (23). It can be recast into a form more like Eq. (23) using the expressions for X_1 and X_3 in Eqs. (15) to eliminate the square roots that appear in Eq. (19).

$$Q = \frac{1}{2}\left[\frac{R_S}{|X_1|} + \frac{R_L}{|X_3|}\left(\frac{R_V - R_2}{R_V}\right)\right]. \qquad (24)$$

In the no-loss case (i.e., $R_2 = 0$), Eq. (24) reduces to Eq. (23).

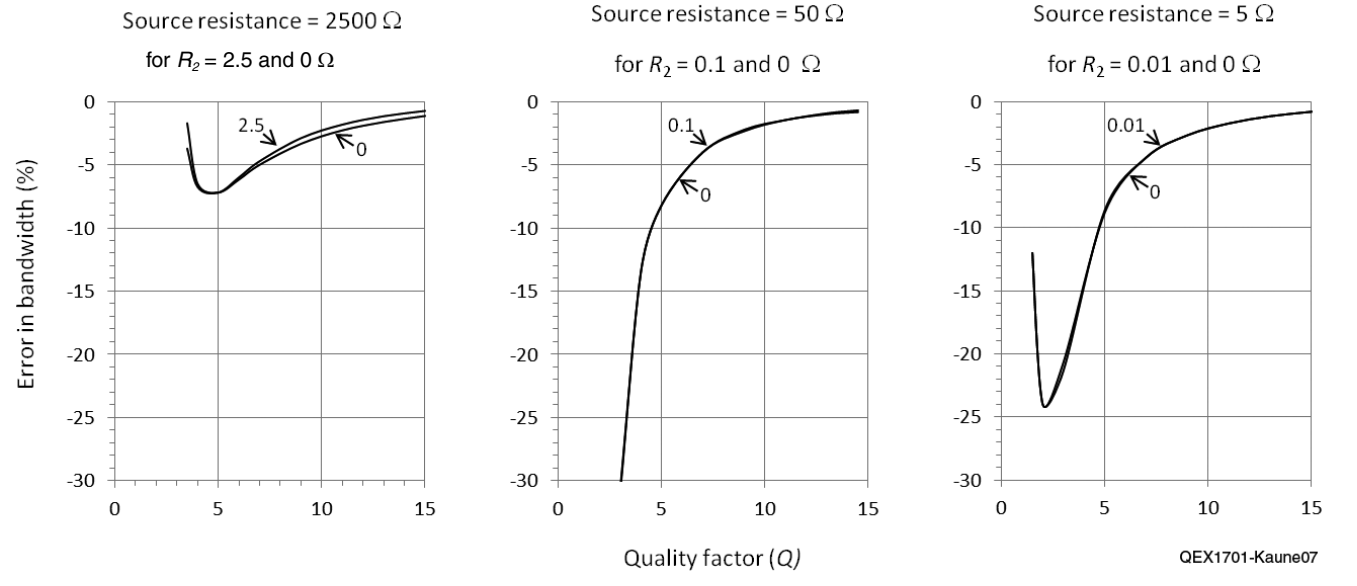

Figure 7 — Error when using calculated quality factor Q of a pi network to estimate its bandwidth, for source resistances of 2500 Ω (left), 50 Ω (center), and 5 Ω (right). The load resistance was 50 Ω in all cases. Curves are for loss resistances of the inductors of 0 Ω and for non-zero values selected to be about in the middle of the range of real inductors.

Is Eq. (24) a good predictor of bandwidth and harmonic distortion? I wrote a computer program (in Visual Basic 2010) that calculated the frequency response of a specified pi network and from this determined bandwidth and harmonic attenuation. Figures 7 and 8 summarize these results.

Figure 7 shows the error resulting from the use of Q, given by Eq. (19), to estimate the bandwidth of a pi network. Graphs are given for a load resistance of 50 Ω and source resistances of 2500, 50, and 5 Ω. In each graph, I selected two values of the loss resistance, one was 0 Ω and the other a value that I thought might be in the middle of the range of loss resistances of actual inductors. For source resistances of 50 and 5 Ω, the curves were not distinguishable. The figure shows that bandwidths estimated using Eq. (19) are always smaller than the actual values. But, for Q values above 5, errors are less than 10% in magnitude.

Pi networks are used not only to match a load to a source but also to attenuate harmonics of the design frequency (i.e., the frequency where the source and load are matched). Figure 8 shows the attenuations of the second, third, and fourth harmonics. As in Figure 7, graphs are given for source resistances of 2500, 50, and 5 Ω. The data in each graph are for the same two loss resistances used in Figure 7; the two curves were indistinguishable. For Q values above 5, attenuation of the second, third, and fourth harmonics were greater than 27 dB.

Specifying Q to Determine R_V

Equation (20) relates the network Q to the virtual resistance, R_V. Values of R_V are limited by the last of Eqs. (17). It is convenient to work with the ratio R_V/R_2 rather than R_V. The limits on this ratio are

$$1 \le \frac{R_V}{R_2} < \text{Min}\left(\frac{R_S}{R_2}, 1+\frac{R_L}{R_2}\right). \qquad (25)$$

Figure 9 shows, for source resistances of 2500, 50, and 5 Ω and a load resistance of

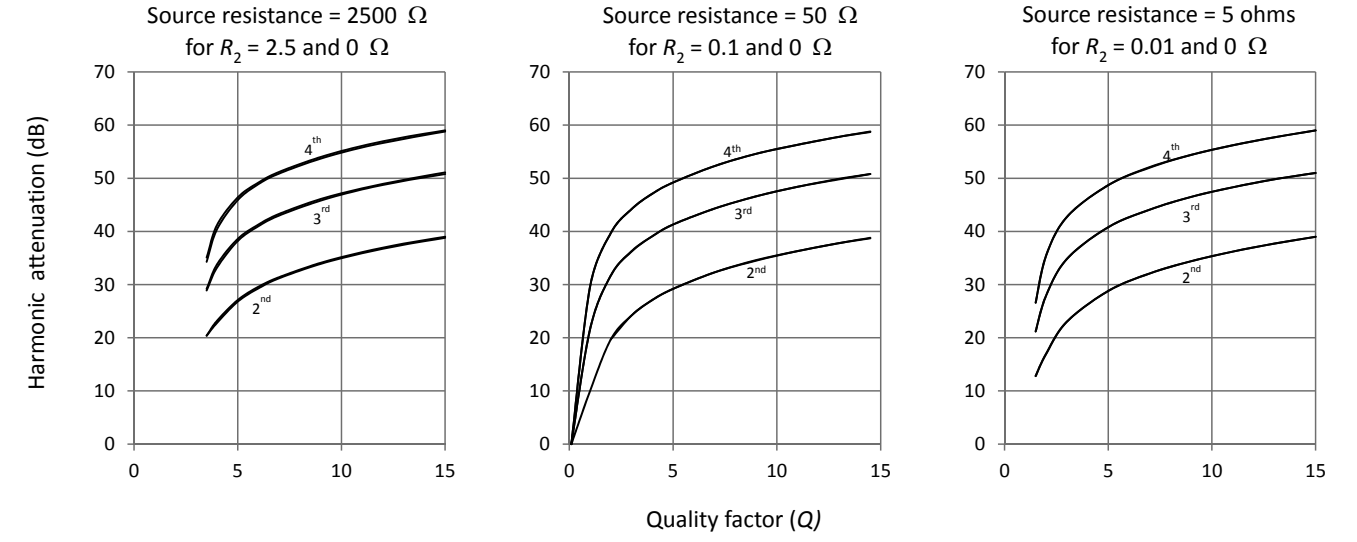
Source resistance = 2500 Ω
for R_2 = 2.5 and 0 Ω
Source resistance = 50 Ω
for R_2 = 0.1 and 0 Ω
Source resistance = 5 ohms
for R_2 = 0.01 and 0 Ω
Harmonic attenuation (dB)
4th
3rd
2nd
Quality factor (Q)

Figure 8 — Harmonic attenuation as a function of quality factor Q of a pi network, for source resistances of 2500 Ω (left), 50 Ω (center), and 5 Ω (right). The load resistance was 50 Ω in all cases. Curves are for loss resistances of the inductors of 0 Ω and for non-zero values selected to be about in the middle of the range of real inductors.

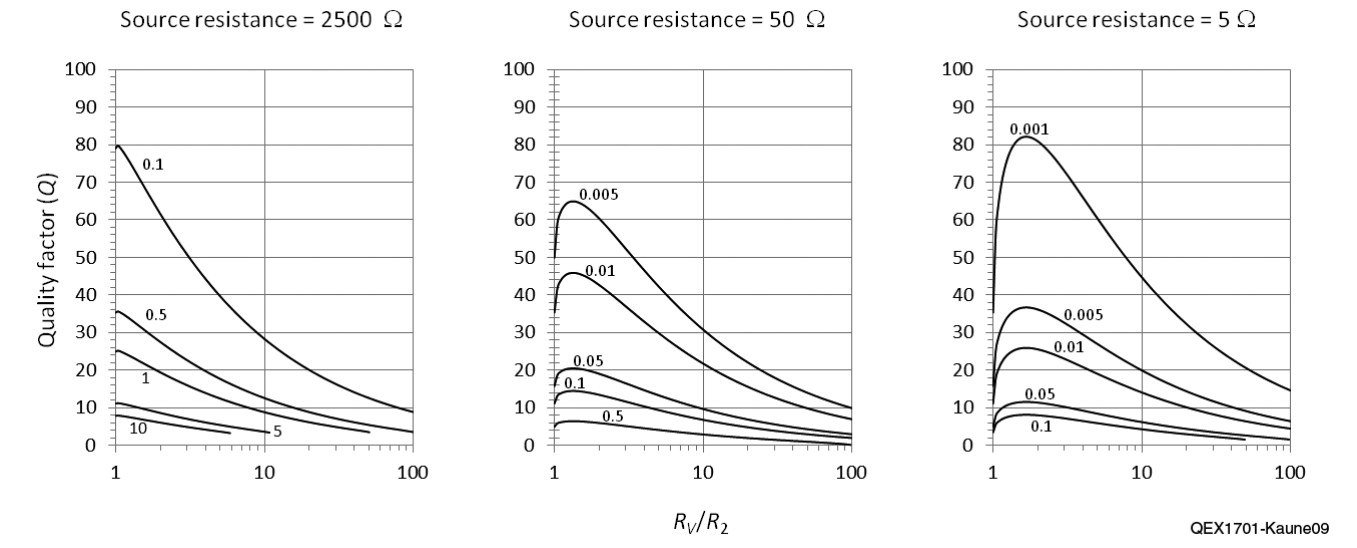

Figure 9 — Quality factor Q as a function of ratio R_V/R_2 for source resistances of 2500 Ω (left), 50 Ω (center), and 5 Ω (right). The load resistance in all cases was 50 Ω. Each curve in a graph is labeled with its respective value of R_2.

50 Ω, the relation between the quality factor of a pi network and R_V/R_2. Curves are shown for various values of the loss resistance, R_2. These data indicate that, at least for the smaller values of R_2, values of Q in excess of 10 can easily be reached. However, these values are obtained as the ratio R_V/R_2 tends towards 1, and loss also increases as R_V/R_2 tends towards 1 as shown by Eq. (23).

The data in Figure 9 can be used to estimate a value for R_V to obtain a specified value of Q. However, it would seem more convenient to have an analytic method for accomplishing this by solving Eq. (19) for R_V as a function of Q. It turns out that inverting Eq. (19) is fairly complicated. Appendix B explains how this solution can be obtained. I do not think it is particularly useful because it seems a better choice is to start with an acceptable level of loss and see what range of Q's can be achieved. The next section addresses this issue.

Specifying Loss to Determine R_V

The relationship between loss, in dB, and R_V is given by Eq. (22). This equation is easily inverted to express R_V as a function of loss.

$$R_V = \frac{R_2}{1-10^{-L_{dB}/10}}. \quad (26)$$

Once loss is specified, R_V and Q are uniquely determined by Eqs. (26) and (19), respectively. Figure 10 shows this relationship for a source impedance of 2500 Ω, 50 Ω, and 5 Ω. The load impedance is 50 Ω in all cases. Each graph contains a family of curves for selected values of the loss resistance R_2 placed on the right edge of each graph.

The graphs in Figure 10 show that there are potentially severe limits on the Q of a pi network if the loss is to be held to a manageable level. For example, suppose we wish to design a pi network whose loss is 1 dB. This means that 20.6% of the power input to the pi network will not appear in the output but is absorbed by R_2, the loss resistance of the inductor. If the transmitter output power is 100 W, this loss might result in warming of the inductor but would probably be acceptable. If the transmitter power is 1 kW, the power dissipation in the inductor would be 206 W, probably

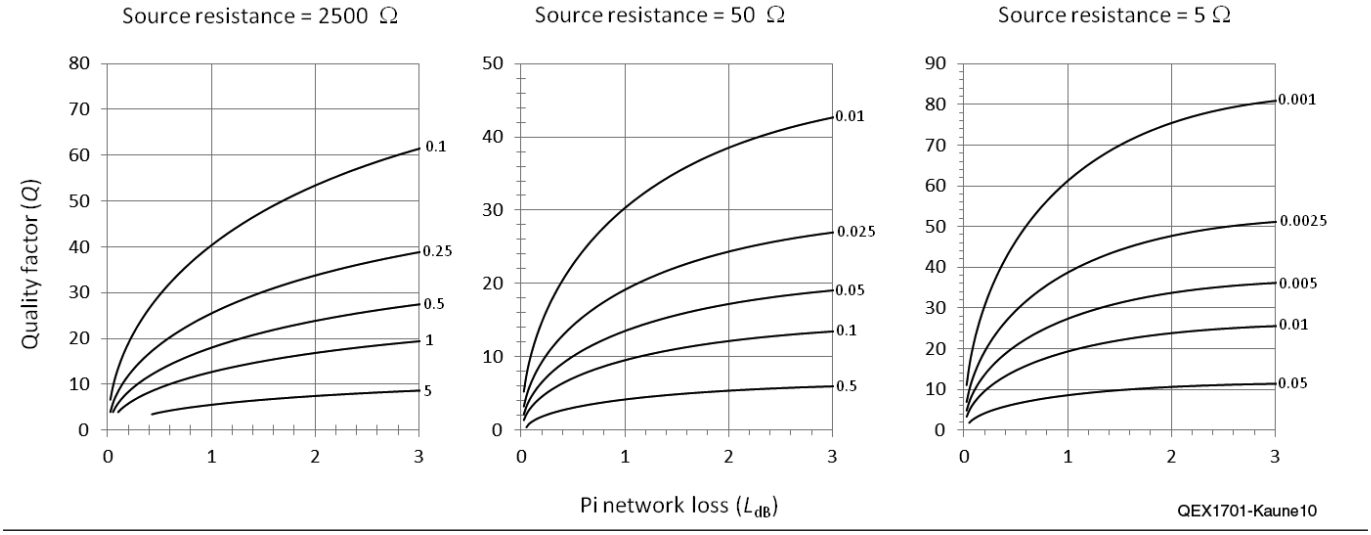

Figure 10 — Quality factor Q as a function of loss in a pi network. Data are shown for source resistances of 2500 Ω (left), 50 Ω (center), and 5 Ω (right). In all cases the load resistance was 50 Ω. Each curve in each graph is labeled with its respective value of R_2.

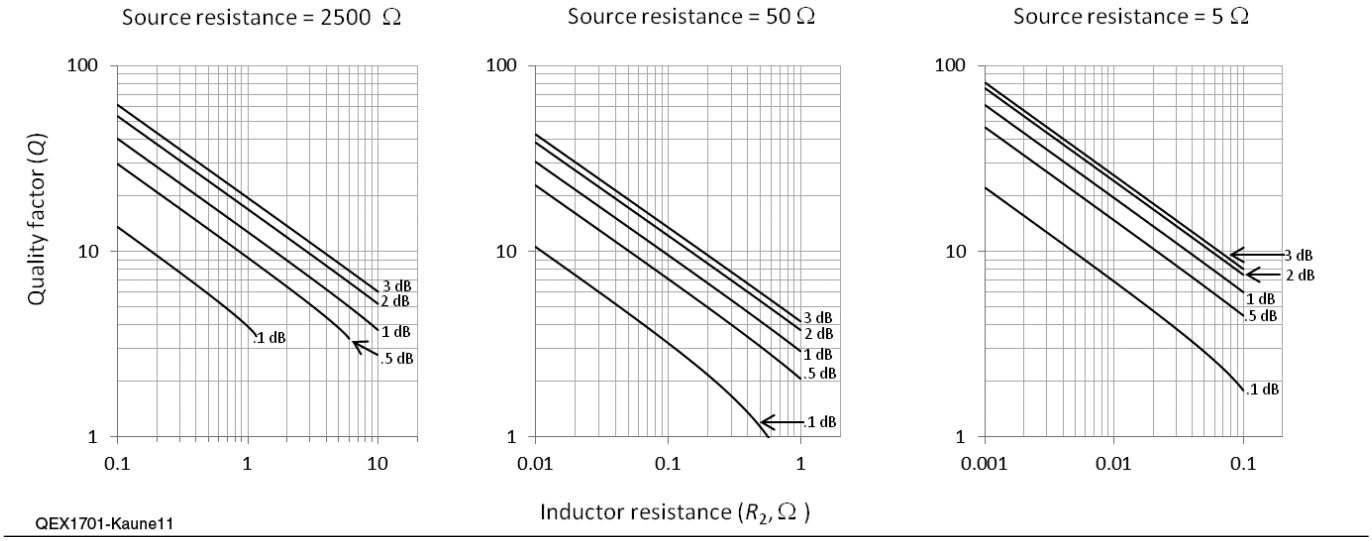

Figure 11 — Quality factor Q as a function of the inductor loss resistance R_2. Data are shown for source resistances of 2500 Ω (left), 50 Ω (center), and 5 Ω (right). In all cases the load resistance was 50 Ω. Each curve in each graph is labeled with its respective value of loss in the pi network.

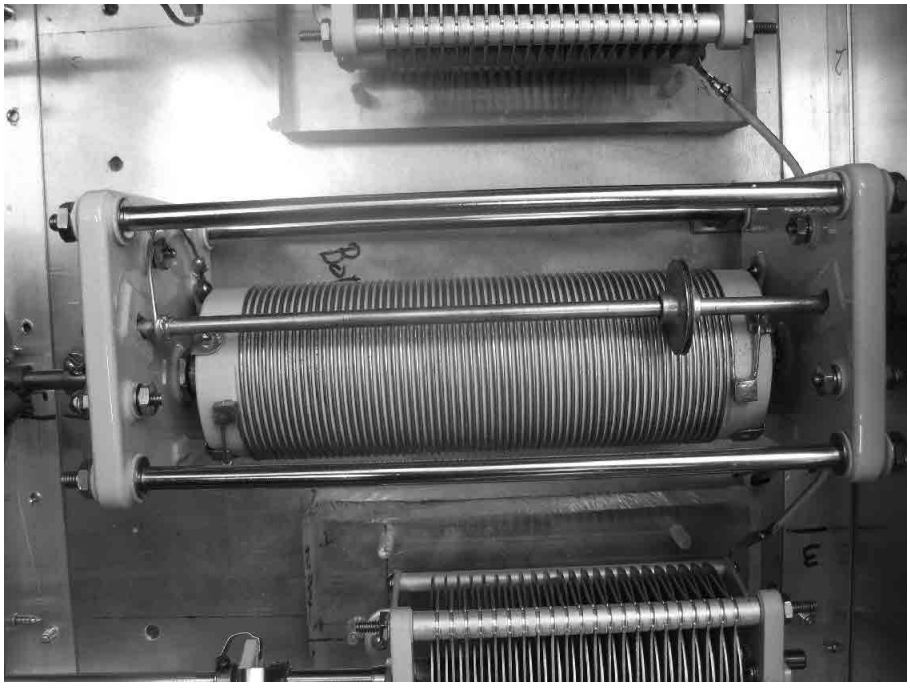

Figure 12 — Variable rotary inductor used in T-network tuner.

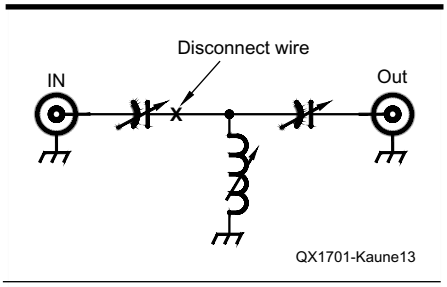

Figure 13 — Schematic diagram of a T network tuner. The wire between the input capacitor and the shunt inductor was disconnected to create a series circuit with the output capacitor and inductor.

unacceptable. But to limit losses to 1 dB and achieve a Q of 10 to 15, a figure that is a typical design goal for transmitters, the data in Figure 10 show that the loss resistance R_2 must be less than about 1 Ω if the source resistance is 2500 Ω, less than about 50 mΩ if the source resistance is 50 Ω, and less than about 20 mΩ for a source resistance of 5 Ω. To achieve losses even less than 1 dB, the loss resistance of the inductor must be substantially less than the values listed in the preceding sentence.

Figure 11 shows the same data in an alternate way that seems quite useful to me. In the three graphs in the figure, Q is plotted

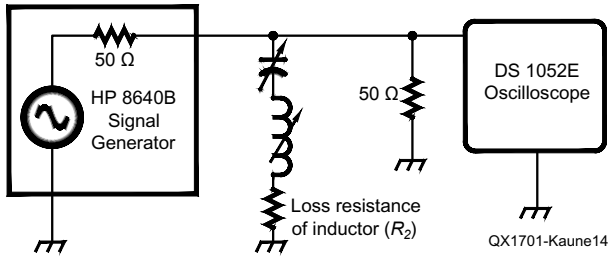

Figure 14 — Experimental setup to measured loss resistance R_2 of inductor shown in Figure 12. The series capacitor and inductor were adjusted to resonance so that their reactances would cancel leaving only R_2 in the circuit.

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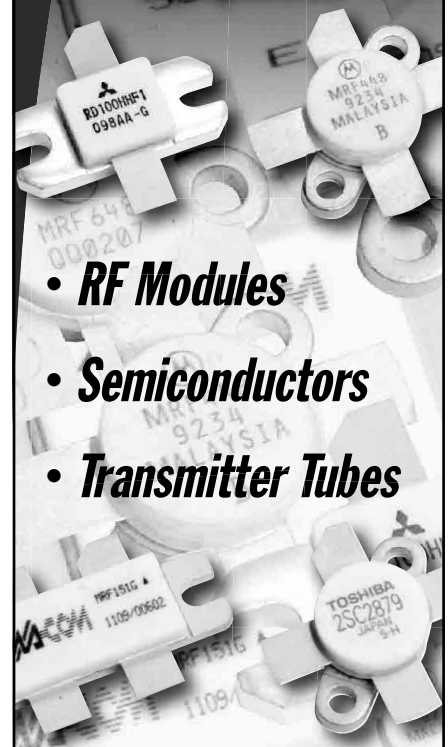

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as a function of the inductor loss resistance, R_2. Families of curves are shown for losses of 0.1, 0.5, 1, 2, and 3 dB. As an example, suppose the source resistance is 2500 Ω and our goal is a pi network with a loss of 0.5 dB (11% of input power dissipated in the inductor) and a Q of 10. The left-hand graph in Figure 11 shows that this can only be accomplished if R_2 is less than about 0.84 Ω.

The work described so far is theoretical. As I progressed, I decided that I needed to obtain some loss data on a real inductor. Therefore, I set out to measure the loss resistance of an actual rotary variable inductor that one might use in a pi network. The next section describes this measurement and the results.

Loss Resistance of a Rotary Variable Inductor

I have a T-network antenna tuner that I built using a rotary variable inductor and two air variable capacitors. Figure 12 shows a photograph of the inductor and part of one of its variable capacitors. Figure 13 is the circuit diagram of a basic T-network tuner. I disconnected the wire between the input capacitor and the shunt inductor; see Figure 13. This change left the output capacitor and the inductor in series with the output connector. I then connected a HP 8640B signal generator to the output connector through a BNC-T. The other side of the T was connected to a Rigol DS1052D digital oscilloscope. Figure 14 shows the complete layout, including the 50 Ω output impedance of the HP signal generator, the 50 Ω input impedance of the oscilloscope (accomplished by placing a 50 Ω terminator at the input of the scope), and the loss resistance R_2 of the inductor. I assume that the loss resistance of the variable capacitor is negligible compared to that of the inductor.

The basic idea is to tune the series capacitor to minimize the voltage measured by the oscilloscope, at which point the reactances of the capacitor and the inductor will cancel each other leaving just R_2, the loss resistance of the inductor. In practice, this is a little more difficult than it might at first sound. First of all, the dip in voltage as resonance is passed through is quite sharp, making it difficult to find the true minimum by adjusting the variable capacitor. I solved this by first adjusting the variable capacitor to as close to the minimum as possible, then using the vernier frequency adjustment on the signal generator to find the true minimum; typically this final adjustment involved a frequency change of no more than about 100 Hz. The second problem stems from the fact that the signal from the signal generator is not a pure sinusoid but contains both fundamental and small harmonic components. The series capacitor-inductor can only be adjusted to null out one frequency, so the harmonic components pass through without attenuation. This problem was solved by using the ability of the Rigol oscilloscope (using its Fourier transform function) to separate a signal into its frequency components and measure the amplitude only of the fundamental.

A measurement occurred in three steps. (1) The inductor was set to a particular value of interest. (2) The series capacitor-inductor were removed from the circuit and the voltage V_1 of the fundamental of the signal noted. (3) The series capacitor-inductor was returned to the circuit and the capacitor and signal generator frequency were adjusted to minimize the fundamental voltage V_2, which was then noted. Referring once again to Figure 14, it is easy to show that

$$V_1 = \frac{V_S}{2}$$
$$V_2 = \frac{R_2}{Z_0 + 2R_2} V_S \quad (27)$$

where Z_0 = 50 Ω. Use the first of these equations to eliminate V_S in the second and then solve the result for the loss resistance R_2 of the inductor. The final equation is

$$R_2 = \frac{Z_0}{2\left(V_1/V_2 - 1\right)}. \quad (28)$$

I took data at 3.75 MHz and 14.2 MHz. The results are in Tables 1 and 2 and show

Table 1.

Measured loss resistances R_2 at a frequency of 3.75 MHz for the inductor shown in Figure 13.

L, μH	*R2*, Ω	*Inductor Q*
24.8	7.49	78
22.1	6.32	82
19.5	4.83	95
16.9	3.88	103
14.4	3.17	107
12.0	2.52	112
9.6	2.01	113
7.4	1.60	109
5.3	1.26	99
4.5	1.15	93
3.8	1.04	86

Table 2.

Measured loss resistances R_2 at 14.2 MHz for the inductor shown in Figure 13.

L, μH	*R2*, Ω	*Inductor Q*
2.5	5.80	38
2.2	4.13	47
1.9	2.96	57
1.6	2.22	65
1.4	1.70	72
1.1	1.36	74
0.92	1.12	74
0.74	0.93	71
0.57	0.79	64
0.43	0.69	56
0.33	0.62	47
0.29	0.61	42
0.26	0.58	39

Table 3.

Example design of a pi network to match a source impedance of 2500 Ω to a load of 50 Ω at a frequency of 3.75 MHz and with a network loss of 0.5 dB. Each row in the table is a successive iteration, obtained from the row before it by using the value for L_2 and the data in Table 1 to calculate an improved estimate of the loss resistance R_2.

Loss, dB	R_2, Ω	C_1, pF	L_2, μH	C_3, pF	Q
0.5	1.00	279	7.21	1917	9.2
0.5	1.57	222	8.97	1442	7.3
0.5	1.89	203	9.80	1267	6.6
0.5	2.05	195	10.18	1193	6.4
0.5	2.13	191	10.37	1159	6.2
0.5	2.17	189	10.46	1142	6.2
0.5	2.19	188	10.50	1134	6.1
0.5	2.20	188	10.52	1130	6.1

Table 4.

Example design of a pi network to match a source impedance of 2500 Ω to a load of 50 Ω at a frequency of 3.75 MHz and with a network loss of 1 dB. Each row in the table is a successive iteration, obtained from the row before it by using the value for L_2 and the data in Table 1 to calculate an improved estimate of the loss resistance R_2.

Loss, dB	R_2, Ω	C_1, pF	L_2, μH	C_3, pF	Q
1.0	1.00	385	5.24	2934	12.7
1.0	1.25	344	5.85	2596	11.3
1.0	1.35	331	6.08	2488	10.9
1.0	1.39	326	6.17	2448	10.8
1.0	1.40	325	6.19	2438	10.7

inductance, the measured loss resistance, and the resulting Q of the inductor ($2\pi fL/R_2$).

I do not know enough about inductors to be able to say whether the data in Tables 1 and 2 are typical of rotary variable inductors. The inductor Q values range from about 78 to 113 at 3.75 MHz and from about 38 to 74 at 14.2 MHz, which are similar to values I have read in various ham-radio-oriented publications.

Designing Pi Networks

All of the analysis presented so far has assumed that the loss resistance R_2 is constant. But, the data in Tables 1 and 2 demonstrate that R_2 varies as the inductance of L_2 varies. This complicates the design process and requires an iterative technique.

Suppose we wish to design a pi network, using my variable rotary inductor, for operation at 3.75 MHz with a loss of 0.5 dB and a Q of about 10. As a first step, the left graph in Figure 10 indicates that the loss resistance must be 1 Ω (or less) to limit loss to 0.5 dB. Thus as a starting point, assume R_2 = 1 Ω. Then use Eq. (26) to calculate R_V, Eqs. (15) to calculate C_1, L_2, and C_3, and Eq. (19) to calculate Q. Results are listed in the first row of Table 3. The calculated inductance is 7.21 µH. Referring to Table 1, we see this inductance is associated with a loss resistance around 1.57 Ω, estimated using linear extrapolation. Now repeat the calculations listed above using R_2 = 1.57 Ω. The second row in the table gives the results. Note that the inductance has increased to a value of 8.97 µH which, according to Table 1, is associated with a loss resistance of 1.89 Ω. The remaining rows in Table 3 give additional iterations, continuing until there is little change from one to the next iteration. The final result shows that with a loss of 0.5 dB, Q = 6.1. This Q is significantly less than our design goal of 10. I find it surprising, and striking, that a loss resistance as small as 2.2 Ω limits so strongly the overall Q of the pi network.

I believe it is generally the case when using a pi network to match the output of a power amplifier to an antenna that one would like to have a Q near 10. To accomplish this, we are evidently going to have to accept a higher loss. Table 4 contains data similar to that in Table 3 but for a loss of 1 dB. After five iterations, we find Q = 10.7. These two examples demonstrate that there is tradeoff between Q and loss. Higher values of Q are associated with larger losses. This is easy to understand. Higher values of Q are a result of a larger storage of energy in the reactive elements of a pi network, and larger stored energy means a larger current circulating between the inductor and the two capacitors. But, this larger current must pass through the loss resistance of the inductor, which leads to larger loss.

Conclusions

Inductor loss plays a substantial role in the design of pi networks. The use of my rotary inductor in a pi network would place substantial constraints on loss and Q: A Q of 10 at 3.75 MHz would require the acceptance of a loss of 1 dB. Lowering loss to 0.5 dB would limit Q to about 6. In designing a pi network, one would want to use a high quality (high Q) inductor. The inductor in Figure 12 is not of sufficient high quality and would not be a good choice for a pi network design.

I developed a computer program that I use to analyze pi networks with inductive loss. This program allows a user to enter source and load resistances, the frequency at which the pi network will be used, the loss resistance R_2, and any one of the following parameters: virtual resistance R_V; Q defined by Eq. (19); Q defined by bandwidth, reactance of C_1, or the network loss in dB. The program calculates the values of C_1, L_2, and C_3, the attenuation of the second through 10th harmonics, and a graph of the frequency response of the network. I would be happy to share this program with interested parties.

APPENDIX A: *Q* of a Cascaded Network

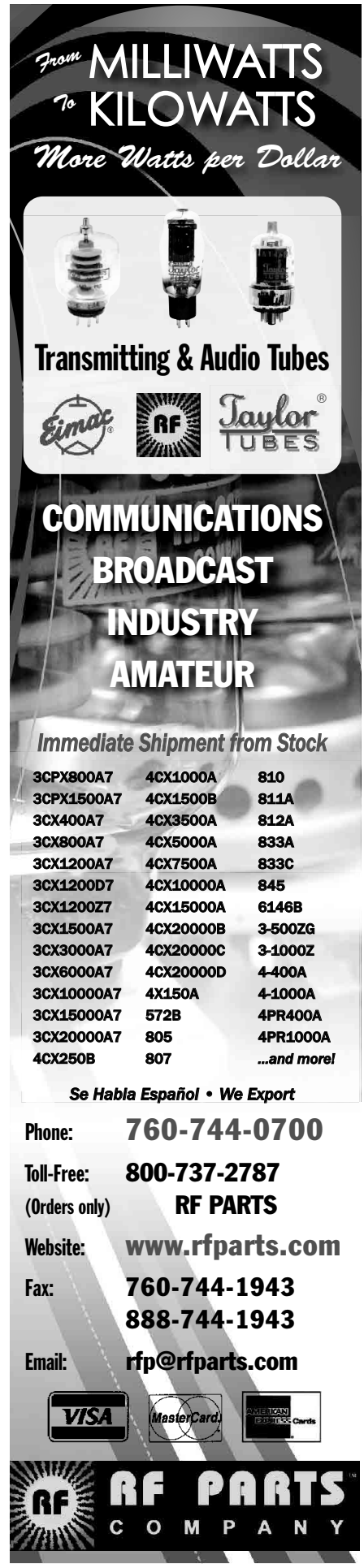
According to Eq. (16), the Q of any network is

$$Q = 2\pi f \frac{W}{D_S + D + D_L}, \qquad \text{(A-1)}$$

where W is the energy stored in the reactive elements, D_S is the power dissipated in the source resistance, D is the power dissipated internally within the network, and D_L is the power dissipated in the load resistance. We assume that the network is used to match a load resistance to a source resistance. This means that the resistance looking into the input of the network will equal the source resistance, which means that the power dissipated in the source resistance will be the same as the power dissipated internally within the network and in the load. That is, $D_S = D + D_L$. Consequently, Eq. (A-1) can be rewritten as

$$Q = 2\pi f \frac{W}{2D_S}. \qquad \text{(A-2)}$$

Now suppose the network can be broken into two cascaded networks, A and B, so that the first network A matches the source resistance R_S to a virtual resistance R_V and the second network B matches R_V to the final load resistance, R_L. Let D_A and D_B be the powers dissipated internally by networks



A and B, respectively, and D_V the power dissipated by the virtual load resistance R_V. Then the quality factors, Q_A and Q_B, for the two networks are

$$Q_A = 2\pi f \frac{W_A}{D_S + D_A + D_V} \quad \text{(A-3)}$$

and

$$Q_B = 2\pi f \frac{W_B}{D_V + D_B + D_L} \quad \text{(A-4)}$$

The source resistance is matched by network A to the virtual load so $D_S = D_A + D_V$. Similarly, the virtual resistance is matched by network B to the load resistance so $D_V = D_B + D_L$. Using these two results, Eqs. (A-3) and (A-4) can be rewritten

$$Q_A = 2\pi f \frac{W_A}{2D_S} \quad \text{(A-5)}$$

and

$$Q_B = 2\pi f \frac{W_B}{2D_V} \quad \text{(A-6)}$$

The energy, W, stored in the entire network is the sum of the energies stored in networks A and B, that is, $W = W_A + W_B$. Using this result in Eq. (A-2), we get

$$Q = 2\pi f \frac{W_A}{2D_S} + 2\pi f \frac{W_B}{2D_S} \quad \text{(A-7)}$$

Now insert Eqs. (A-5) and (A-6) into (A-7). The result is

$$Q = Q_A + \frac{D_V}{D_S} Q_B \,. \quad \text{(A-8)}$$

The ratio D_V/D_S is just the efficiency, ε_A, of network A to deliver power to its load. The final result is, then,

$$Q = Q_A + \varepsilon_A Q_B \quad \text{(A-9)}$$

In the pi network case treated in this paper, the loss occurred only in network B so ε_A=1

APPENDIX B: Calculate R_V from Q

The calculation of R_V from a specification of Q starts with Eq. (19) which expresses Q as a function of R_V. This equation is

$$Q = \frac{1}{2}\left[\sqrt{\frac{R_S - R_V}{R_V}} + \sqrt{\left(\frac{R_V - R_2}{R_V}\right)\left(\frac{R_L + R_2 - R_V}{R_V}\right)}\right]. \quad \text{(B-1)}$$

Inverting this equation involves considerable algebra. It must be squared twice to eliminate the square roots. The result is a polynomial equation in R_V:

$$AR_V^4 + BR_V^3 + CR_V^2 + DR_V + E = 0 \quad \text{(B-2)}$$

where

$$\left.\begin{aligned} A &= 16Q^2\left(Q^2+1\right) \\ B &= -8Q^2\left(R_S + 2R_2 + R_L\right) \\ C &= \left(R_L + 2R_2 - R_S\right)^2 + 8Q^2R_2\left(R_L + R_2\right) \\ D &= -2R_2\left(R_L + R_2\right)\left(R_L + 2R_2 - R_S\right) \\ E &= R_2^2\left(R_L + R_2\right)^2 \end{aligned}\right\} \quad \text{(B-3)}$$

There are four roots to equations like (B-2). To calculate these roots, I used a method on a website.[4] Some (or all) of the roots may be complex numbers that we can reject immediately. If there are two or four real roots, some may not satisfy Eq. (B-1) and can be rejected. Others may not satisfy the inequality listed last in Eqs. (15). If there are two remaining roots, select the larger of the two because it will involve less loss in the network.

Consider this example: R_S = 2500 Ω, R_L = 50 Ω, R_2 = 2 Ω, and Q = 10. Use Eqs. (B-3) to calculate the coefficients A through E, and calculate the roots of Eq. (B-2) using the method mentioned above. The four solutions for R_V are 7.74 Ω, 4.98 Ω, (-0.0411 + j 0.00683) Ω, and (-0.0411 – j 0.00683) Ω. The latter two are complex and can be discarded. When placed in Eq. (B-1), the second solution yields a value for Q of 12.4, not the target value of 10. The first solution, 7.74 Ω, yields Q of 10 and is the desired value.

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Notes

[1]Bill Kaune, W7IEQ, "Quality Factor, Bandwidth and Harmonic Attenuation of Pi Networks", *QEX*, Sep 2015, p. 2.
[2]*The ARRL Handbook Book, 2016 Edition.* ARRL item no. 0413, available from your ARRL dealer, or from the ARRL Store, Telephone toll-free in the US 888-277-5289, or 860-594-0355, fax 860-594-0303; **www.arrl.org/shop/**; **pubsales@arrl.org**, ARRL, Newington, CT.
[3]Alan Payne, "Measuring the Loss in Variable Air Capacitors", **g3rbj.co.uk**, 2013.
[4]**jwilson.coe.uga.edu/EMAT6680Fa09/Davenport/Solving%20Quartic%20Equations.pdf**